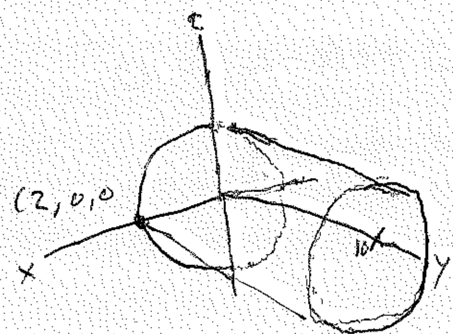
Ch. 7 Surface integrals of real-valued functions and vector fields

Sec 7.1-7.2 Parametric surfaces

- Recall: A curve in $\mathbb{R}^3$ is parametrized by $\vec{c}(t) = (x(t), y(t), z(t))$.
- Def A parametric surface is $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$.

Example $\vec{c}(u,v) = (2\cos u, v, 2\sin u)$, $u \in [0, 2\pi]$, $v \in [0, 10$



= Side of cylinder (not front or back)

Example We saw in Ch. 1 that $\vec{r}(u,v) = \vec{r}_0 + u\vec{a} + v\vec{b}$, $u \in \mathbb{R}$, $v \in \mathbb{R}$ is the plane through $\vec{r}_0$ spanned by $\vec{a}$ and $\vec{b}$.

Example The sphere $x^2 + y^2 + z^2 = r^2$ can be parametrized by

$$\vec{r}(x,y) = \begin{cases} (x, y, \sqrt{r^2 - x^2 - y^2}), & z \geq 0 \\ (x, y, -\sqrt{r^2 - x^2 - y^2}), & z \leq 0 \end{cases}$$

or in spherical coordinates by

$$\vec{r}(u,v) = (r\sin u\cos v, r\sin u\sin v, r\cos v), \ u \in [0, 2\pi], v \in [0, \pi]$$

Example In general, $z = f(x,y)$ can be parametrized by

$$\vec{r}(x,y) = (x, y, f(x,y)).$$

- Tangent plane to a parametric surface S parametrized by $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$

Set $\vec{r}_u(u,v) = \left(\frac{\partial x}{\partial u}(u,v), \frac{\partial y}{\partial u}(u,v), \frac{\partial z}{\partial u}(u,v)\right)$

$\vec{r}_v(u,v) = \left(\frac{\partial x}{\partial v}(u,v), \frac{\partial y}{\partial v}(u,v), \frac{\partial z}{\partial v}(u,v)\right)$

Then $\vec{n} = \vec{r}_u \times \vec{r}_v$ is a normal vector to S. Thus,

- Tangent plane equation: $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

or

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

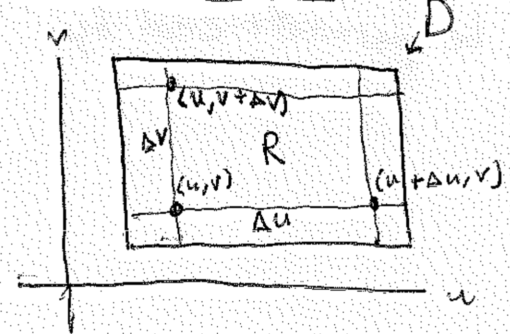
where $\vec{r}(u_0, v_0) = (x_0, y_0, z_0)$ is the point of tangency

$\vec{n} = \vec{r}_{u_0} \times \vec{r}_{v_0} = (a, b, c)$

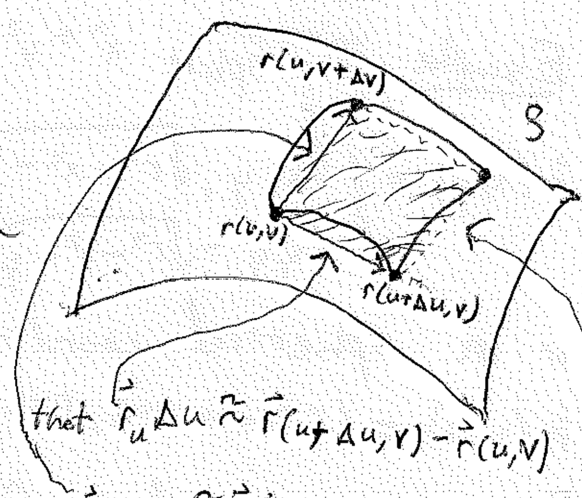
$\vec{r} = (x, y, z)$

Sec 7.3 Surface integrals of real-valued functions

Surface area



$\xrightarrow{r(u,v)}$ maps R to the patch shown



From $\vec{r}_u \approx \dfrac{r(u+\Delta u, v) - r(u,v)}{\Delta u}$, we see that $\vec{r}_u \Delta u \approx \vec{r}(u+\Delta u, v) - \vec{r}(u,v)$

Similarly, $\vec{r}_v \Delta v \approx \vec{r}(u, v+\Delta v) - \vec{r}(u,v)$

Recall that $\|r_u \Delta u \times r_v \Delta v\| = \|r_u \times r_v\| \Delta u \Delta v$

= area of parallelogram spanned by $\vec{r}_u \Delta u$ and $\vec{r}_v \Delta v$.

Thus we can approximate the surface area of S with such parallelograms:

$$\text{surface area of } S \approx \sum_{i=1}^{n}\sum_{j=1}^{n} \|\vec{r}_u(u_i,v_j) \times \vec{r}_v(u_i,v_j)\| \underbrace{\Delta u_i \Delta v_j}_{\Delta A_{ij}}$$

$$\longrightarrow \iint_D \|\vec{r}_u \times \vec{r}_v\| \, dA \quad \text{as } n \longrightarrow \infty.$$

- Def $\text{Surface area of } S = A(S) = \iint_D \|\vec{r}_u \times \vec{r}_v\| \, dA.$

- Def For S a smooth, C^1 surface parametrized by $\vec{r}: D \to S$, $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$, where D is type 1/2 region in $\mathbb{R}^2$, and a continuous function $f: S \to \mathbb{R}$, the surface integral of f over S is

$$\boxed{\iint_S f \, dS = \iint_D f(\vec{r}(u,v)) \, \|\vec{r}_u \times \vec{r}_v\| \, dA.}$$

- Remarks: - compare this with the path integral of a real-valued function:

$$\int_{\vec{c}} f \, ds = \int_a^b f(\vec{c}(t)) \, \|\vec{c}\,'(t)\| \, dt$$

- in the case that $\vec{r}(x,y) = (x, y, g(x,y))$, we have

$$\iint_S f \, dS = \iint_D f(x,y,g(x,y)) \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1} \; dA.$$

- Note $\boxed{dS = \|\vec{r}_u \times \vec{r}_v\| \, dA}$

<u>Example</u> Surface area of sphere of radius r

<u>Soln</u>

$$\vec{r}(\theta,\phi) = (r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\phi),\ \theta\in[0,2\pi],\ \phi\in[0,\pi]$$

$$\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\phi\sin\theta & r\sin\phi\cos\theta & 0 \\ r\cos\phi\cos\theta & r\sin\phi\sin\theta & -r\sin\phi \end{vmatrix}$$

$$= (-r^2\sin^2\phi\cos\theta,\ -r^2\sin^2\phi\sin\theta,\ -r^2\sin\phi\cos\phi)$$

and

$$\|\vec{r}_\theta \times \vec{r}_\phi\| = \sqrt{r^4\sin^4\phi\cos^2\theta + r^4\sin^4\phi\sin^2\theta + r^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{r^4\sin^4\phi + r^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{r^4\sin^2\phi}$$

$$= r^2\sin\phi,$$

thus

$$A = \iint_D \|\vec{r}_\theta \times \vec{r}_\phi\|\, dA = \int_0^{2\pi}\int_0^{\pi} r^2\sin\phi\, d\phi\, d\theta$$

$$= r^2\int_0^{2\pi} \Big[-\cos\phi\Big]_0^{\pi} d\theta$$

$$= r^2\int_0^{2\pi} 2\, d\theta = \boxed{4\pi r^2}$$ as expected.

<u>Example</u> $\iint_S x^2\, dS$, where S = unit sphere.

<u>Soln</u> $\vec{r}(\theta,\phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, $\theta\in[0,2\pi]$, $\phi\in[0,\pi]$

We saw above that $\|\vec{r}_\theta \times \vec{r}_\phi\| = 1^2\sin\phi$, so

$$\iint_S x^2\, dS = \iint_D (\sin\phi\cos\theta)^2 \cdot \sin\phi\ dA$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\phi\cos^2\theta\, d\phi\, d\theta = \longrightarrow$$

$$= \left(\int_0^{2\pi} \cos^2 \theta \, d\theta \right) \left(\int_0^{\pi} \sin^3 \phi \, d\phi \right)$$

$$= \left(\frac{1}{2} \int_0^{2\pi} 1 + \cos 2\theta \, d\theta \right) \left(\int_0^{\pi} \sin \phi \left(1 - \cos^2 \phi\right) d\phi \right)$$

$$= \left(\frac{1}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{2\pi} \right) \left(\int_0^{\pi} \sin \phi - \underbrace{\sin \phi \cos^2 \phi \, d\phi}_{u\text{-sub}} \right)$$

$$= \left(\pi \right) \left(\left[-\cos \phi + \frac{1}{3} \cos^3 \phi \right]_0^{\pi} \right)$$

$$= \boxed{\frac{4}{3} \pi}.$$

<u>Theorem</u> If S is parametrized by smooth, C^1 parametrizations $\vec{r}(u,v) : D \to \mathbb{R}^3$ and $\vec{r}^*(u^*, v^*) : D^* \to \mathbb{R}^3$, then

$$\iint_S f \, dS = \iint_D f(\vec{r}(u,v)) \, \| \vec{r}_u \times \vec{r}_v \| \, dA = \iint_{D^*} f(\vec{r}^*(u^*, v^*)) \, \| \vec{r}^*_u \times \vec{r}^*_v \| \, dA^*$$

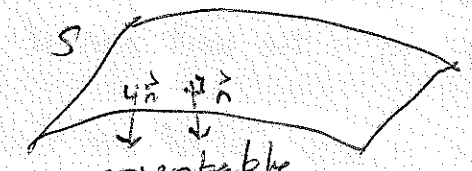
In words, surface integrals <u>of real-valued functions</u> are independent of parametrization.

<u>Sec 7.4</u> Surface integrals of vector fields

We need to discuss orientation before defining these.

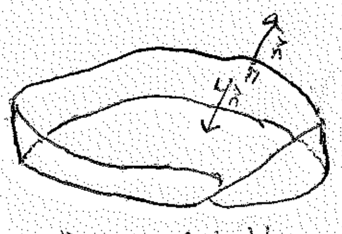
• <u>Def</u> A surface S is <u>orientable</u> if it has two sides. If this is the case, making a choice for normal vectors to point up or down gives S an <u>orientation</u>.

orientable
<u>upward</u> orientation



orientable
<u>downward</u> orientation

Möbius strip: An ant/normal vector can crawl around surface without crossing an edge and end up pointing the opposite way it started.

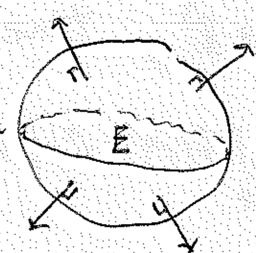


There is really just one side, no inside/outside.

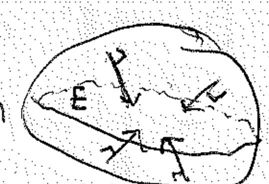
nonorientable

Def For a <u>closed surface</u>, one that is the boundary of a solid region E, the <u>positive orientation</u> is the one for which normal vectors point outward from E, and inward pointing normal vectors give the <u>negative orientation</u>.

positive orientation



negative orientation



Def For S a smooth, C^1 surface parametrized by $\vec{r} : D \to S$, $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$, where D is a type 1/2 region in $\mathbb{R}^2$, and a continuous vector field $F : S \subseteq \mathbb{R}^3 \to \mathbb{R}^3$, the <u>surface integral of F over S</u> is

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{N} \, dS = \iint_D F(r(u,v)) \cdot (r_u \times r_v) \, dA = \iint_D F \cdot \vec{n} \, dA}$$

where $\vec{n}$ is the normal vector $\vec{n} = \vec{r}_u \times \vec{r}_v$, and $\vec{N}$ is the unit normal vector $\vec{N} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$.

- This is also called the <u>flux of F across S</u>.

If F is the velocity vector field of a fluid, $\iint_S F \cdot d\vec{S} =$ net volume of fluid that flows across S per unit time.

Example $\iint_S F \cdot d\vec{S}$, where $F = (z, y, x)$ and

S is the unit sphere.

Soln Let $r(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, $\phi \in [0,\pi]$, $\theta \in [0, 2\pi]$

By a previous example,

$\vec{r}_\phi \times \vec{r}_\theta = (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi)$

Next, $F(\vec{r}(\phi,\theta)) = (\cos\phi, \sin\phi\sin\theta, \sin\phi\cos\theta)$,

and $F(\vec{r}(\phi,\theta)) \cdot (\vec{r}_\phi \times \vec{r}_\theta) = 2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta$

$$\text{So } \iint_S F \cdot d\vec{S} = \iint_D F \cdot (r_\phi \times r_\theta)\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi} 2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta \, d\phi\, d\theta$$

$$= \left(2\int_0^{\pi}\sin^2\phi\cos\phi\, d\phi\right)\underbrace{\left(\int_0^{2\pi}\cos\theta\, d\theta\right)}_{=0} + \left(\int_0^{\pi}\sin^3\phi\, d\phi\right)\left(\int_0^{2\pi}\sin^2\theta\, d\theta\right)$$

$$= 0 + \left(\int_0^{\pi}\underbrace{\sin^3\phi}_{=\sin\phi(1-\cos^2\phi)}\, d\phi\right)\left(\int_0^{2\pi}\underbrace{\sin^2\theta}_{=\frac{1}{2}(1-\cos 2\theta)}\, d\theta\right)$$

$$= \boxed{\frac{4\pi}{3}}$$

- Theorem Let S be an oriented surface, $\vec{r}: D \to \mathbb{R}^3$, $\vec{r}^*: D^* \to \mathbb{R}^3$ two C^1 parametrizations of S with normal vectors $\vec{n}$ and $\vec{n}^*$. Then

$$\iint_D F \cdot \vec{n}\, dA = \begin{cases} \iint_{D^*} F \cdot \vec{n}^*\, dA^*, & \text{if } \vec{r} \text{ and } \vec{r}^* \text{ have the same orientation} \\ -\iint_{D^*} F \cdot \vec{n}^*\, dA^*, & \text{if } \vec{r} \text{ and } \vec{r}^* \text{ have opposite orientations} \end{cases}$$

- Note This is in contrast with surface integrals of real-valued functions.